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Theory of 'self-similarity' of periodic approximants to a quasilattice: II. The case of the dodecagonal quasilattice

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Abstract. The space groups of the periodic approximants (PAs) to the dodecagonal quasilattice in two dimensions are investigated. The Bravais lattices of the PAs are tetragonal (p4mm), rectangular (pmm), hexagonal (p6mm) and rhombic (cmm). We present several series of PAs, each of which is derived from a prototype PA by a successive application of the deflation-and-rescaling (DAR), and the space group is common among the members. Each series is composed of two subseries; the consecutive members in each subseries are related by the 'proper' DAR and the two subseries are related to each other by the 'improper' DAR.

1. Introduction

We have shown in a previous paper (Niizeki 1991c, to be referred to as I) that self-similarity of a quasilattice (QL) gives rise to a striking relationship among the periodic approximants (PAs) to the QL: the PAs are grouped into series so that (i) each series is generated from its prototype by a successive application of the deflation-and-rescaling, (ii) the space group is common among the members of the series and (iii) the unit cell of the PA is scaled up by τ with the series number, where τ is the scale of self-similarity of the relevant QL.

We have excluded in I the case of the dodecagonal quasilattice (DQL) in two dimensions (2D) because its self-similarity is unique among important QLs in 2D and 3D: the relevant self-similarity transformation is 'improper' in the sense that it is not a pure dilatation but a combined operation of a dilatation and a rotation through $\pi/12$ (Stampfli 1986, Niizeki and Mitani 1987). Moreover, the DQL is rich in space groups of its PAs in comparison with the octagonal QL or the decagonal one because (i) the cyclic group 12 has four non-trivial crystallographic subgroups, 2, 3, 4 and 6, whereas 8 or 10 has only two or one, and (ii) $\sqrt{3}$ has three series of best approximants. We shall investigate in this paper the space groups of the PAs to the DQL and the relationships among them derived from the self-similarity.

We summarize in section 2 the properties of the DQL, and in section 3 its self-similarity. In section 4 we investigate extensively PAs to the DQL. This section is divided into four subsections. Section 4.1 is an extension of the theory in I to the DQL. Fibonacci number analogues associated with the DQL are introduced in section 4.2. The mother lattices of the PAs to the DQL are investigated in section 4.3 and several important series of PAs are presented in section 4.4. Section 5 is devoted to a discussion.

2. The dodecagonal quasilattice

A QL is obtained by the projection method from a mother lattice which is a periodic lattice with higher dimensionality than the physical dimension (see, for example, Janssen 1988). Prior to the projection, the mother lattice is cut with a strip which is characterized by a phase vector and the window (a finite domain in the internal space). Two QLs with different phase vectors but a common window are locally isomorphic but two QLs with different windows are not.

A DQL is obtained when the mother lattice L is the dodecagonal lattice (p12mm) in 4D (Niizeki 1989a). The 4D Euclidean space E_4 into which L is embedded is decomposed into the physical space E_2 and the internal one E_2'; $E_4 = E_2 \oplus E_2'$. Let $\boldsymbol{\varepsilon}_i = (\boldsymbol{e}_i, \boldsymbol{e}_i')$ with $\boldsymbol{e}_i \in E_2$ and $\boldsymbol{e}_i' \in E_2'$ be the basis vectors of L: $L = \{\Sigma_i n_i \boldsymbol{\varepsilon}_i | (n_1, n_2, n_3, n_4) \in \boldsymbol{Z}^4\}$. Then $\boldsymbol{e}_i$ (or $\boldsymbol{e}_i'$) are related to each other by $\boldsymbol{e}_{i+1} = r\boldsymbol{e}_i$ (or $\boldsymbol{e}_{i+1}' = r'\boldsymbol{e}_i'$) with $i = 1, 2$ and 3, where r (or r') is the rotation through $\pi/6$ (or $-5\pi/6$); $r' = -r$. $|\boldsymbol{e}_i|$ (or $|\boldsymbol{e}_i'|$) take a common value, which we shall denote by a (or a'). In fact, the value of a' is nothing to do with the properties of the DQL and we can choose a' arbitrarily. $\boldsymbol{e}_i$ (or $\boldsymbol{e}_i'$) are linearly independent over $\boldsymbol{Z}$. Note that $\boldsymbol{e}_1$ and $\boldsymbol{e}_4$ are perpendicular to each other and so are $\boldsymbol{e}_1'$ and $\boldsymbol{e}_4'$.

The $\boldsymbol{Z}$-module $L_P \equiv PL = \{\Sigma_i n_i \boldsymbol{e}_i | (n_1, n_2, n_3, n_4) \in \boldsymbol{Z}^4\}$ (or $L_P' \equiv P'L$) with P (or P') being the projector onto E_2 (or E_2') is a dense set in E_2 (or E_2') and called a prequasilattice. E_2 has an incommensurate orientation with respect to L and $L \cap E_2 = \{0\}$.

Twelve vectors $\boldsymbol{e}_i \equiv (r)^{i-1}\boldsymbol{e}_1$ (or $\boldsymbol{e}_1' \equiv (r')^{i-1}\boldsymbol{e}_1'$), $i = 1$–12, represent the vertex vectors of a regular dodecagon D (or D'), whose point group is $G = 12$mm. The 4D rotation $\hat{r} \equiv r \oplus r'$ is an element of the point group $\hat{G}$ ($\simeq$12mm) of L. $\hat{G}$ is generated by $\hat{r}$ and the 4D mirror $\hat{\sigma} = \sigma \oplus \sigma'$, where σ (or σ') is a 2D mirror whose axis includes $\boldsymbol{e}_1$ (or $\boldsymbol{e}_1'$). $\hat{G}$ leaves E_2 and E_2' invariant and acts onto these two subspaces as 2D point groups G and G' ($\simeq G$). We sometimes identify $\hat{G}$ with G. L_P is invariant against G.

We may write $\hat{r}(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2\boldsymbol{\varepsilon}_3\boldsymbol{\varepsilon}_4) = (\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2\boldsymbol{\varepsilon}_3\boldsymbol{\varepsilon}_4)\boldsymbol{R}$, where $\boldsymbol{R}$ is a unimodular matrix given in the appendix. From $\hat{r}^6 = -1$ and $\hat{r}^{12} = 1$ we obtain $\boldsymbol{R}^6 = -I$ and $\boldsymbol{R}^{12} = I$ (the unit matrix is denoted in this paper by I irrespective of its dimensionality).

We can index $\boldsymbol{x} \equiv \Sigma_i x_i \boldsymbol{\varepsilon}_i \in E_4$ as $[x_1x_2x_3x_4]$. We may say $\boldsymbol{x}$ is a rational point with respect to L if x_i are all rationals. Rationality of a point in E_2 (or E_2') with respect to L_P (or L_P') is defined similarly. Then, P (or P') is a bijection between the set of all the rational points in E_4 and that in E_2 (or E_2'). If $\boldsymbol{x} \in E_4$ is a rational point, we may index $P\boldsymbol{x}$ and $P'\boldsymbol{x}$ with the same index as that of $\boldsymbol{x}$.

The DQL obtained from L with the projection method is written (Niizeki 1991b) as

$$Q(\boldsymbol{x}, W) = \{P(\boldsymbol{l}+\boldsymbol{x}) | \boldsymbol{l} \in L, P'(\boldsymbol{l}+\boldsymbol{x}) \in W\} \tag{1}$$

where $\boldsymbol{x} \in E_4$ is the 4D phase vector and W ($\subset E_2'$) the window. We assume that W is a polygon with the point symmetry G' ($\simeq$12mm) and its vertices are rational points with respect to L_P'.

Several types of DQLs are obtained by choosing different windows (Niizeki and Mitani 1987). The canonical window W_c of the DQL is a regular dodecagon whose vertices are given by $\boldsymbol{v}_i = (r')^{i-1}\boldsymbol{v}_1$, $i = 1$–12, with $\boldsymbol{v}_1 = [1\bar{1}1\bar{1}]/3 \in E_2'$. W_c includes D' ($W_c \supsetneqq D'$). The relevant DQL is formed of the vertices of Stampfli's dodecagonal quasiperiodic tiling (Stampfli 1986), as shown in figure 1. The basic tiles of the tiling are a square, a regular triangle and a rhombus. On the other hand, if $W_{D'} \equiv D'$ is used as the window, we obtain a DQL associated with a tiling with squares, triangles and

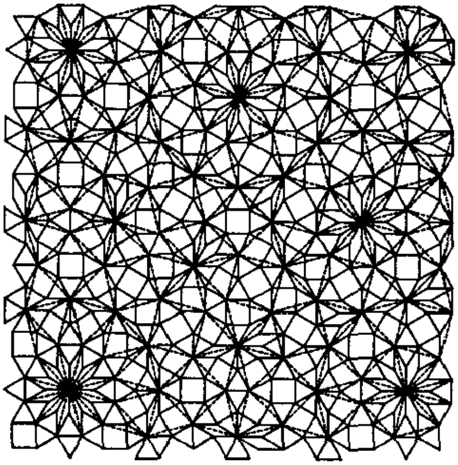


Figure 1. The dodecagonal quasilattice obtained from the 4D dodecagonal lattice L by using the canonical window. The lattice points are given by the positions of the vertices of the dodecagonal quasiperiodic tiling. The centres of squares, triangles or rhombi are derived from the SPs of type M, T or C of L. The once inflated QL is superimposed with the broken lines. The bonds in the inflated QL have different directions from those of the original QL.

trigonal hexagons; a hexagon is a union of one square, two triangles and one rhombus. Let W_s be the dodecagonal star whose convex (or concave) vertices coincide with the vertices of W_c (or $W_{D'}$). Then $W_{D'} \subsetneq W_s \subsetneq W_c$ and the DQL derived with W_s is obtained from that with $W_{D'}$ by dividing each trigonal hexagon into four basic tiles.

$x \in E_4$ is called a special point (SP) if its point symmetry with respect to L is a centring group, which is a subgroup of $\hat{G}$. An SP is a rational point. There exist six classes of SPs (Niizeki 1989e, 1990). The six are represented by the symbols Γ, X, C, M, T and T'. Their representatives are $[0000]$, $[h000]$, $[hh00]$, $[h0h0]$, $[t0t0]$ and $[tttt]$ with $h = \frac{1}{2}$ and $t = \frac{1}{3}$ and their point groups are 12mm, mm, mm, 4mm, 3m and 3m, respectively.

The projection of an SP of L onto E_2 (or E_2') is called an SP with respect to L_P (or L_P'). The vertices of W_c are SPs of type T'. The vertices of the dodecagonal tiling in figure 1 are derived from Γ, the midpoints of the bonds from X and the centres of rhombi, triangles or squares from C, T or M.

The 12 mirrors of 12mm are grouped into two classes, Δ and Σ (Niizeki 1991a); a mirror of type Δ passes a vertex of D, while that of type Σ the midpoint of an edge. A representative of Δ (or Σ) is σ (or $r\sigma$).

3. Self-similarity of the DQL

The 4D transformation $\hat{\tau} = \tau I \oplus \tau' I$ with $\tau = 2+\sqrt{3}$ being a PV unit and $\tau' = 2-\sqrt{3}$ $(=1/\tau)$, the algebraic conjugate of τ, induces a unimodular transformation among ε_i:

$$\hat{\tau}(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4) = (\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)M \tag{2}$$

where M is a unimodular matrix given in the appendix; $\det(M) = 1$. It follows that $\hat{\tau}L = L$. Note that $\hat{\tau} = 2 + \hat{r} + \hat{r}^{-1}$, so that $M = 2I + R + R^{-1}$. $\hat{\tau}$ acts as a scale transformation onto each of the two subspaces E_2 and E_2' and is commutable with $\hat{G}$. Using these results, we can prove that the DQL has a self-similarity whose scale is equal to $\tau = 2+\sqrt{3}$ (Niizeki 1989a).

Since $\boldsymbol{R}^2 + I$ is a regular matrix, we can conclude from $\boldsymbol{R}^6 + I = 0$ that $\boldsymbol{R}^4 - \boldsymbol{R}^2 + I = 0$ and, consequently, $(\boldsymbol{R}+\boldsymbol{R}^2)(\boldsymbol{R}-\boldsymbol{R}^2) = I$. Therefore, $\boldsymbol{M}_0 \equiv \boldsymbol{R}+\boldsymbol{R}^2$ is a unimodular matrix, which is given in the appendix ($(\boldsymbol{M}_0)^{-1} = \boldsymbol{R}-\boldsymbol{R}^2$), and $\hat{\tau}_0 \equiv \hat{r}+\hat{r}^2$ as well as $\hat{\tau}$ is an automorphism of L; $\hat{\tau}_0 L = L$. We may write $\hat{\tau}_0 = \tau_0 \oplus \tau_0'$ with $\tau_0 = r + r^2$ and $\tau_0' = r' + (r')^2$. We obtain $\tau_0' = -(\tau_0)^{-1}$ because $r' = -r$ and $r^4 - r^2 + 1 = 0$. On the other hand, τ_0 is equal to $\tau_{\mathrm{p}} r_8$, where $\tau_{\mathrm{p}} \equiv (\sqrt{3}+1)/\sqrt{2}$ is the 'platinum ratio' and r_8 the rotation through $\pi/4$. That is, τ_0 (or $\tau_0' = -(\tau_0)^{-1}$) is an expanding (or contracting) similarity transformation of E_2 (or E_2'). It cannot be reduced to a pure dilatation because $r_8 \notin \hat{G}$; τ_0 is an 'improper' dilatation. $\hat{r}$, $\hat{\tau}$ and $\hat{\tau}_0$ are commutable with each other and we obtain $(\hat{\tau}_0)^2 = \hat{r}^3 \hat{\tau}$. Since $\hat{r}^3 \in \hat{G}$, $(\hat{\tau}_0)^2$ is equivalent to $\hat{\tau}$ as a transformation of L. Note that $\hat{\tau}_0$ exchanges the two types of mirrors, Δ and Σ (Niizeki 1991a).

Using these results we can prove as in I that

$$Q(\boldsymbol{x}, W) = \tau_0^{-1} Q(\hat{\tau}_0 \boldsymbol{x}, \tau_0' W). \tag{3}$$

Since $\tau_0' W \subsetneq W$, $Q(\hat{\tau}_0 \boldsymbol{x}, \tau_0' W)$ is a sublattice (subset) of $Q(\hat{\tau}_0 \boldsymbol{x}, W)$, so that $\tau_0 Q(\boldsymbol{x}, W) \subsetneq Q(\hat{\tau}_0 \boldsymbol{x}, W)$. It follows that the DQL has the 'improper' self-similarity with the transformation τ_0 (Niizeki 1989a). The 'improper' inflation of the DQL in figure 1 is superimposed in the same figure. The directions of the bonds of the inflated DQL bisect those of the original ones because $15° \equiv 45° \bmod 30°$. It is important in a later argument that the vertex vectors of the dodecagon $\tau_0' D'$ are given by $\boldsymbol{e}_1' + \boldsymbol{e}_{i+1}'$, $i = 1$–12, with $\boldsymbol{e}_{13}' \equiv \boldsymbol{e}_1'$.

The 4D transformation $\hat{\rho} \equiv \rho I \oplus \rho' I$ with $\rho = 1+\sqrt{3}$ and $\rho' = 1-\sqrt{3}$ satisfies $\hat{\rho} L \subsetneq L$. More precisely, $\det(\hat{\rho}) = 4$, and $\hat{\rho} L$ is one of the four equivalent sublattices into which L is divided (Niizeki 1989c). The integer matrix associated with ρ is given by $\boldsymbol{M} - I$ because $\rho = \tau - 1$. ρ is a (non-unit) PV number because $|\rho'| < 1$ and we can prove that the DQL has so-called the type II self-similarity with scale ρ (Niizeki 1989c). A similar argument applies to the non-unit PV number $3 + 2\sqrt{3}$ $(= \sqrt{3}\tau)$.

4. The periodic approximants to the DQL

4.1. The general theory

A PA to the DQL is constructed by the projection method from a 4D lattice $\tilde{L}$ which is a deformation of L due to a linear phason strain. The strain makes a 2D lattice plane Π_2 of L overlap perfectly with the physical space E_2; $\Phi\Pi_2 = E_2$ and $\tilde{L} = \Phi L$, where Φ is the linear transformation associated with the phason strain. We shall call $\tilde{L}$ a commensurately deformed lattice (CDL) because it is fully commensurate with E_2, in contrast to L.

Π_2 is spanned by two lattice vectors of L, so that it is indexed by a 4×2 integer matrix $\boldsymbol{K}$ whose columns index the two lattice vectors. Π_2 is indexed, equivalently, by the dual index $\boldsymbol{J}$, which is a 2×4 integer matrix satisfying $\boldsymbol{JK} = 0$ and $\operatorname{rank}(\boldsymbol{J}) = 2$. Let $\hat{H}$ be a maximal subgroup of $\hat{G}$ such that it leaves Π_2 invariant. Then it is the point group of $\tilde{L}$ and is decomposed as $H \oplus H'$ in which H acts onto E_2 and H' onto E_2'. In fact, $H \simeq H'$ and H is crystallographic in 2D. $\hat{H}$ is identified with H.

Every SP of $\tilde{L}$ has its associate among the SPs of L. If an SP of L has inversion symmetry, it remains as an SP after the phason strain is introduced, while SPs of the types T and T' remain as SPs only when the strain preserves the trigonal symmetry.

The basis vectors $\tilde{\varepsilon}_i$ $(=\Phi\varepsilon_i)$ of $\tilde{L}$ are decomposed as $\tilde{\varepsilon}_i=(e_i, \tilde{e}'_i)$. Only two of the $\tilde{e}'_i$s are linearly independent over $\boldsymbol{Z}$, $(\tilde{e}'_1\tilde{e}'_2\tilde{e}'_3\tilde{e}'_4)\boldsymbol{K}=0$, and we obtain

$$(\tilde{e}'_1\tilde{e}'_2\tilde{e}'_3\tilde{e}'_4)=(\boldsymbol{b}_1\boldsymbol{b}_2)\boldsymbol{J} \tag{4}$$

where $\boldsymbol{b}_i\in E'_2$ are linearly independent over $\boldsymbol{R}$. Two rows of $\boldsymbol{J}$ represent rational approximations to the incommensurate ratios associated with the two directions $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$.

The Bravais lattice of the PAs obtained from $\tilde{L}$ is given by $L_2=\tilde{L}\cap E_2$, while $L_s\equiv P'L$ is called the shadow lattice of $\tilde{L}$ (Niizeki 1991b). Two vectors defined by

$$(\boldsymbol{a}_1\boldsymbol{a}_2)=(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4)\boldsymbol{K} \tag{5}$$

are linearly independent over $\boldsymbol{R}$ and belong to L_2. $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (or $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$) are basis vectors of L_2 (or L_s) only when $\boldsymbol{K}$ (or $\boldsymbol{J}$) is 'unimodular'. This case will be called irreducible. The point groups of L_2 and L_s are given by H and H'.

A PA to the DQL (1) *is given by*

$$\tilde{Q}(\tilde{x}, \tilde{W})=\{P(\boldsymbol{l}+\tilde{x})\,|\,\boldsymbol{l}\in\tilde{L}, P'(\boldsymbol{l}+\tilde{x})\in\tilde{W}\} \tag{6}$$

with $\tilde{x}=\Phi x$ and $\tilde{W}$ being an appropriate deformation of W. The Bravais lattice of $\tilde{Q}$ is equal to L_2 and the space group is determined by the relative position of $P'\tilde{x}$ to L_s (Niizeki 1991a). In particular, the point group is equal to that of $P'\tilde{x}$ with respect to L_s. Moreover, if $\tilde{x}$ is an SP of $\tilde{L}$, $P\tilde{x}$ is an SP of $\tilde{Q}$ (Niizeki 1991b). $\tilde{Q}$ is called a regular PA if its point symmetry conforms to the Bravais lattice L_2. In order to obtain a regular PA, it is necessary that $P'\tilde{x}$ is located on an SP or a special line of L_s (Niizeki 1991b). The space group of a PA associated with a special line of L_s is a subgroup of that associated with an SP which is located on the special line (Niizeki 1991b).

Let $v\in E'_2$ be a vertex of W and assume that v is indexed as $[\xi_1\xi_2\xi_3\xi_4]$ with ξ_i being rationals. Then it is natural to assume that $\tilde{W}$ is a polygon and $\tilde{v}=\Sigma_i\,\xi_i\tilde{e}'_i$ is the corresponding vertex of $\tilde{W}$ to v. This prescription determines $\tilde{W}$ uniquely (cf I). We shall denote it symbolically as $\tilde{W}=\Phi W$. For example, the vertex vectors of the dodecagon $\tilde{D}'\equiv\Phi D'$ are given by $\tilde{e}'_i$.

The 2D lattice plane Π_2 is transformed by $\hat{\tau}_0$ to another one, $\Pi'_2=\hat{\tau}_0\Pi_2$, which is indexed by $K'=KM_0$ or $J'=-J(M_0)^{-1}$. The CDL associated with Π'_2 is given by $\tilde{L}'=\hat{\tau}_0\tilde{L}$ and we obtain $L'_2\equiv\tilde{L}'\cap E_2=\tau_0L_2$ and $L'_s\equiv P'\tilde{L}'=\tau'_0L_s$; L'_2 and L'_s are similar to L_2 and L_s, respectively. The point group of $\tilde{L}'$ is given by $\hat{\tau}_0\hat{H}(\hat{\tau}_0)^{-1}$, which is isomorphic with $\hat{H}$. $\tilde{L}'$ is obtained from L with the phason strain $\Phi'=\hat{\tau}_0\Phi(\hat{\tau}_0)^{-1}$, which is smaller than Φ.

We shall denote by $\tilde{Q}'(y, V)$ the PA which is derived from $\tilde{L}'$ with an arbitrary phase vector y and a window V. Then we can prove in a similar way as in I that

$$\tilde{Q}'(\tilde{x}', \tau'_0\tilde{W})=\tau_0\tilde{Q}(\tilde{x}, \tilde{W}) \tag{7}$$

with $\tilde{x}'=\hat{\tau}_0\tilde{x}$. On the other hand, $\tilde{Q}'(\tilde{x}', \tilde{W}')$ with $\tilde{W}'=\Phi'W$ is a PA to $Q(\hat{\tau}_0x, W)$ because $\tilde{x}'=\Phi'\hat{\tau}_0x$. Moreover, $\tilde{Q}'(\tilde{x}', \tau'_0\tilde{W})\subsetneq\tilde{Q}'(\tilde{x}', \tilde{W}')$ provided that $\tau'_0\tilde{W}\subsetneq\tilde{W}'$. Thus we can conclude that $\tilde{Q}'(\tilde{x}', \tilde{W}')$ is the DAR of $\tilde{Q}(\tilde{x}, \tilde{W})$.

Let $\tilde{W}'_c=\Phi'W_c$ and assume that $\tilde{v}'_i$ are the vertex vectors of $\tilde{W}'_c$. Then we can show that the vertex vectors of the dodecagon $\tau'_0\tilde{W}_c$ are given by $\tilde{v}'_i+\tilde{v}'_{i+1}$, $i=1$–12, with $\tilde{v}'_{13}\equiv\tilde{v}'_1$. A similar relation remains correct also in the case of the window $W_{D'}$ or W_s. $\tau'_0\tilde{W}$ is invariant against the point group which is the restriction of the point group of $\tilde{L}'$ onto E'_2. Therefore, $\tilde{Q}'(\tilde{x}', \tilde{W}')$ and $\tilde{Q}'(\tilde{x}', \tau'_0\tilde{W})$ have a common space group. Using these results and making a similar reasoning as that in I we can conclude that the PAs to the DQL are grouped *into several series, each of which is derived from the prototype* PA in it by a successive application of the deflation and rescaling (DAR) and the space group is common among the PAs in the series.

A series of PAs, $\tilde{Q}_0$, $\tilde{Q}_1$, $\tilde{Q}_2, \ldots$, can be divided into two subseries $\tilde{Q}_0$, $\tilde{Q}_2$, $\tilde{Q}_4, \ldots$ and $\tilde{Q}_1$, $\tilde{Q}_3$, $\tilde{Q}_5, \ldots$ and two consecutive members in each subseries are related by the 'proper' DAR, which is defined by using the transformation $\hat{\tau}$; the second subseries is derived from the first by the 'improper' DAR.

4.2. *The Fibonacci number analogues associated with $2+\sqrt{3}$*

The quadratic irrational $\tau=2+\sqrt{3}$ is the positive root of the equation $\tau^2=4\tau-1$. Iterating the equality $\tau=4-1/\tau$ yields an infinite continued fraction, though it is not regular. It gives rise to a series of rational approximants to τ, which are the ratios of consecutive numbers of the integer series defined by the recursion relation $u_{k+1}=4u_k-u_{k-1}$ with $u_0=0$ and $u_1=1$; early members of the series $\{u_k\}$ is listed in table 1. From the recursion relation, we can prove that $u_{k+1}-u_k/\tau=\tau^k$, so that $u_{k+1}-u_k/\tau'=(\tau')^k$ or, equivalently, $u_{k+1}-u_k\tau=1/\tau^k$, which gives a measure of the accuracy of the approximant $\tau\approx u_{k+1}/u_k$. Note that $u_{k+1}/u_k>\tau$.

Table 1. The Fibonacci like series associated with $2+\sqrt{3}$.

$\{v_k\}=\{1, 1, 3, 11, 41, \ldots\}$
$\{u_k\}=\{0, 1, 4, 15, 56, \ldots\}$
$\{w_k\}=\{1, 2, 7, 26, 97, \ldots\}$

Let us derive from the series $\{u_k\}$ another two, $\{v_k\}$ and $\{w_k\}$, by $v_k\equiv u_k-u_{k-1}$ and $w_k\equiv u_k+v_k$ (see table 1). The new series are generated by the same recursion relation as that of $\{u_k\}$ but with different initial conditions. The parity alternates in $\{u_k\}$ and $\{w_k\}$, while $\{v_k\}$ is composed of odd numbers. Note that $v_{k+1}-v_k/\tau=(\sqrt{3}-1)\tau^k$ and $w_{k+1}-w_k/\tau=\sqrt{3}\tau^k$.

Best approximants to τ are obtained by the continued fraction theory from its regular continued fraction expansion. They are grouped into three series $\{u_{k+1}/u_k\}$, $\{v_{k+1}/v_k\}$ and $\{w_{k+1}/w_k\}$; u_{k+1}/u_k and v_{k+1}/v_k are principal convergents to the continued fraction but w_{k+1}/w_k is an intermediate one between u_{k+1}/u_k and v_{k+2}/v_{k+1}. Note that $\{v_{k+1}/v_k\}$, $\{w_{k+1}/w_k\}<\tau$. Since $w_{k-1}<v_k<u_k<w_k<v_{k+1}$ for $k\geqslant 2$, the three series of approximants are merged into one grand series and they are the members of the three-cycles in the grand series.

If p/q is a best approximant to τ, $(p-2q)/q$ $(=p/q-2)$ is to $\sqrt{3}$. Note, in this respect, that $u_{k+1}-2u_k=w_k$ and $w_{k+1}-2w_k=3u_k$. That is, w_k/u_k, t_k/v_k and $3u_k/w_k$ are best approximants to $\sqrt{3}$ with $t_k\equiv v_{k+1}-2v_k$ $(=u_k+u_{k-1})$. Note that $w_k+\sqrt{3}u_k=\tau^k$ and $t_k+\sqrt{3}v_k=(\sqrt{3}-1)\tau^k$.

Let us assume that p/q is an approximant to $\sqrt{3}$. Then $\tau(p+\sqrt{3}q)=p'+\sqrt{3}q'$ with

$$\begin{pmatrix} p' \\ q' \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}. \tag{8}$$

p'/q' is a more accurate approximant to $\sqrt{3}$ than p/q; p'/q' is the next generation to

p/q with respect to the scaling with τ. For example, if $p/q = w_k/u_k$, then $p'/q' = w_{k+1}/u_{k+1}$. Similarly, the 'next generation' to p/q with respect to the scaling with ρ $(=1+\sqrt{3})$ is given by $(p+3q)/(p+q)$. For example, the 'next generation' in this sense to w_k/u_k is t_{k+1}/v_{k+1}.

4.3. *The mother lattices of the* PAs *with mirrors*

We shall investigate only the PAs having two mirrors perpendicular to each other. The mirrors must be of the same type (Δ or Σ). We consider for the moment the case of type Δ mirrors, which are assumed to be parallel to $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ in E_2. Let us take the Cartesian coordinate systems for E_2 and E_2' so that the two axes coincide with the two mirrors. Then we may write

$$(\boldsymbol{e}_1'\boldsymbol{e}_2'\boldsymbol{e}_3'\boldsymbol{e}_4') = \begin{pmatrix} 2 & -\sqrt{3} & 1 & 0 \\ 0 & -1 & \sqrt{3} & -2 \end{pmatrix} \tag{9}$$

with $a' = 2$. The first (or second) component of $\boldsymbol{e}_i'$ refers to the horizontal (or vertical) mirror in E_2', so that $\sqrt{3}$ in the first (or second) row in (9) is the incommensurate ratio associated with the relevant mirror axes. We take a 2D lattice plane Π_2 indexed by the dual index

$$\boldsymbol{J} = \begin{pmatrix} 2q & -p & q & 0 \\ 0 & -s & r & -2s \end{pmatrix} \tag{10}$$

where p/q (or r/s) is a rational approximant to $\sqrt{3}$ in the first (or second) row in (9). The dual index $\boldsymbol{K}$ to $\boldsymbol{J}$ is given by

$${}^t\boldsymbol{K} = \begin{pmatrix} p & 2q & 0 & -q \\ -s & 0 & 2s & r \end{pmatrix} \tag{11}$$

which satisfies $\boldsymbol{JK} = 0$. We can assume that p/q and r/s are simple fractions. Then $\boldsymbol{J}$ and $\boldsymbol{K}$ are both irreducible (unimodular) except for the case where $p \equiv s \bmod 2$ and $q \equiv r \bmod 2$ but they are both reducible in the exceptional case.

The CDL associated with Π_2 is characterized by the pair of fractions $\langle p/q, r/s\rangle$. Note that $\langle r/s, p/q\rangle$ is equivalent to $\langle p/q, r/s\rangle$ because they are transformed to each other by the 4D mirror $\hat{r}^3\hat{\sigma}$ $(\in \hat{G})$. We are interested in the case where both p/q and r/s are best approximants to $\sqrt{3}$.

From (5) and (11) we can conclude that $\boldsymbol{a}_1$ (or $\boldsymbol{a}_2$) is horizontal (or vertical) and $a_1 \equiv |\boldsymbol{a}_1| = a(p+\sqrt{3}\,q)$ and $a_2 \equiv |\boldsymbol{a}_2| = a(r+\sqrt{3}\,s)$. Similarly, $\boldsymbol{b}_1$ (or $\boldsymbol{b}_2$) is horizontal (or vertical) and $b_1 \equiv |\boldsymbol{b}_1| = \sqrt{3}\,a'/(p+\sqrt{3}\,q)$ and $b_2 \equiv |\boldsymbol{b}_2| = \sqrt{3}\,a'/(r+\sqrt{3}\,s)$. If $\boldsymbol{K}$ (or $\boldsymbol{J}$) is irreducible, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (or $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$) are the basis vectors of L_2 (or L_s); the Bravais class to which L_2 (or L_s) belongs is p4mm (a square lattice) or pmm (a rectangular lattice). On the contrary, if it is reducible, the centring takes place and the basis vectors are $\boldsymbol{a}_1' = (\boldsymbol{a}_1 - \boldsymbol{a}_2)/2$ and $\boldsymbol{a}_2' = (\boldsymbol{a}_1 + \boldsymbol{a}_2)/2$ (or $\boldsymbol{b}_1' = \boldsymbol{b}_1 - \boldsymbol{b}_2$ and $\boldsymbol{b}_2' = \boldsymbol{b}_1 + \boldsymbol{b}_2$); L_2 (or L_s) belongs to p6mm (a hexagonal lattice) or cmm (a rhombic lattice).

The 2D lattice plane $\hat{\tau}\Pi_2$ of L is indexed by $\boldsymbol{K}' = \boldsymbol{KM}$ (or $\boldsymbol{J}' = \boldsymbol{JM}^{-1}$). $\boldsymbol{K}'$ (or $\boldsymbol{J}'$) takes the form (11) (or (10)) but p, q, r and s are replaced by their next generations, p', q', r' and s' (cf (8)). Consequently, $\hat{\tau}\langle p/q, r/s\rangle = \langle p'/q', r'/s'\rangle$.

We consider next the effect of the transformation $\hat{\tau}_0$ onto a CDL. Since $\hat{\tau}_0$ exchanges the two types of mirrors, Δ and Σ, it changes the type of mirrors of a CDL to the other type. Therefore, a CDL with type Σ mirrors is written with an appropriate CDL $\langle p/q, r/s\rangle$ as $\hat{\tau}_0\langle p/q, r/s\rangle$, which we shall denote as $\langle p/q, r/s\rangle_\Sigma$. Then, the relevant lattice plane of L is indexed by $\boldsymbol{K}_\Sigma \equiv \boldsymbol{K}\boldsymbol{M}_0$ (or $\boldsymbol{J}_\Sigma \equiv -\boldsymbol{J}(\boldsymbol{M}_0)^{-1}$). We obtain

$$^t(\boldsymbol{K}_\Sigma) = \begin{pmatrix} q & t & t & q \\ -u & -v & v & u \end{pmatrix} \qquad \boldsymbol{J}_\Sigma = \begin{pmatrix} t & -q & -q & t \\ v & -u & u & -v \end{pmatrix}$$

where $t = p+q$, $u = r+2s$ and $v = r+s$. Therefore, the two mirrors of $\langle p/q, r/s\rangle_\Sigma$ are parallel to $\boldsymbol{e}_2+\boldsymbol{e}_3$ and $\boldsymbol{e}_3-\boldsymbol{e}_2$ in E_2. The expression for $\boldsymbol{J}_\Sigma$ is natural because the components of $\boldsymbol{e}'_i$, $i = 1$-4, along the mirror axis $\boldsymbol{e}'_2+\boldsymbol{e}'_3$ (or $\boldsymbol{e}'_3-\boldsymbol{e}'_2$) are proportional to $(\rho, -1, -1, \rho)$ (or $(-\rho', -1, 1, \rho')$) with $\rho = 1+\sqrt{3}$ (or $\rho' = 1-\sqrt{3}$) and t/q (or v/u) is an approximant to ρ (or $|\rho'|$).

Since a PA associated with $\langle p/q, r/s\rangle_\Sigma$ is constructed with $\langle p/q, r/s\rangle$ by (7), we need not consider the CDL $\langle p/q, r/s\rangle_\Sigma$ any further. It should be noted, however, that PAs of type Δ and those of type Σ have no relations in the case of the octagonal QL (see I) or the decagonal one (Niizeki 1991b).

4.4. *Several important series of PAs*

We will not be interested in a PA such that the values of the two lattice constants a_1 and a_2 are very different. We consider in this section the case where p/q is fixed to w_k/u_k and r/s takes one of the four choices: (I) w_k/u_k, (II) t_k/v_k, (III) $3u_k/w_k$ and (IV) w_{k-1}/u_{k-1}. That is, a_1 is fixed to $a\tau^k$ and $a_2/a_1 = 1, \sqrt{3}-1, \sqrt{3}$ and $1/\tau$, respectively. On the basis of the parity sequences in $\{u_k\}$, $\{v_k\}$ and $\{w_k\}$ together with $t_k \equiv v_k \bmod 2$, we can show easily that K and J are irreducible in I and II but are reducible in III and IV. More precisely, L_2 and L_s belong both to p4mm, pmm, p6mm and cmm for I, II, III and IV, respectively. The unit cell of L_2 in IV is similar to the rhombic tile in the DQL. Since $\langle p/q, r/s\rangle$ and $\langle p/q, r/s\rangle_\Sigma$ are distinguished, there exist eight series of CDLs, IΔ, IΣ, IIΔ, IIΣ, IIIΔ, IIIΣ, IVΔ and IVΣ. A CDL in each of the eight series can be designated, alternatively, by the series symbol and the number in the series, e.g. IΔ_k, IΣ_k, IIΔ_k, etc.

IΔ and IΣ, for example, can be considered to be subseries of the union series, $I \equiv$ I$\Delta \cup$ IΣ, which is generated from IΔ_0 by a successive application of $\hat{\tau}_0$, while IΔ (or IΣ) is from IΔ_0 (or IΣ_0) by a successive application of $\hat{\tau}$.

We consider only regular approximants associated with the SPs of L_s. The relevant SPs are Γ ([00]) and M ([hh] with $h = \frac{1}{2}$) for the square lattice, Γ, M, X ([h0]) and Y ([0h]) for the rectangular lattice, Γ and T ([21]/3) for the hexagonal lattice and Γ and M for the rhombic lattice. The point group of each of these SPs is the same as that of L_s except for the case of T, whose point group is 3m. A PA associated with an SP will be represented by the symbol for the SP as $\tilde{Q}[\Gamma]$, $\tilde{Q}[X]$, etc, or, more precisely, as I$\Delta_k[\Gamma]$, II$\Sigma_k[X]$, etc.

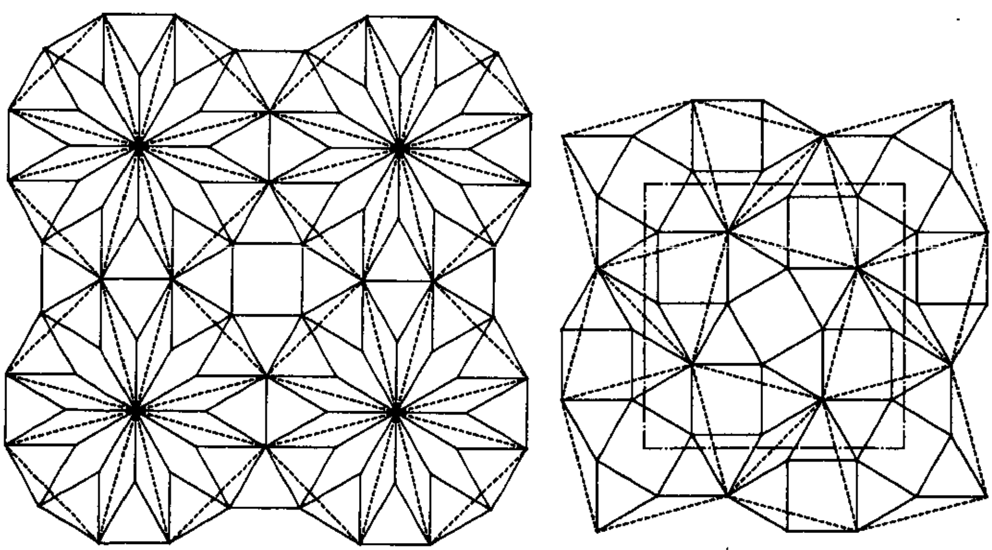
The SPs of class C of the 4D dodecagonal lattice L have mirrors of type Σ only and the mirrors are lost by the introduction of the phason strain of type Δ; only the inversion symmetry is preserved on the deformation. On the other hand, the SPs of classes X, M and R have mirrors of type Δ. These mirrors are lost by the phason strain unless they are parallel to the mirrors of the strain. Thus, CDLs in IΔ, for example, have only two classes of SPs with the point group 4mm; they are derived from Γ and M of L.

The space group of $\tilde{Q}[\Gamma]$ is always identical to that of L_2. However, a few considerations are necessary for the cases of other SPs. We begin by considering the case $\tilde{Q}[M] = \mathrm{I}\Delta_k[M]$, whose point group is 4mm. A PA with the space group p4mm has two classes of SPs whose point groups are 4mm. Such SPs must be derived from SPs of classes Γ and M of $\mathrm{I}\Delta_k$. However, the latter SPs are projected onto the SPs of $\mathrm{I}\Delta_k[\Gamma]$ only. Therefore, the space group of $\mathrm{I}\Delta_k[M]$ cannot be p4mm and is determined to be p4g. The space group may be explicitly shown as $\mathrm{I}\Delta_k[M]$/p4g. By similar arguments, we can determine the space groups of PAs of other cases. These results are summarized in table 2.

Several PAs with different space groups are presented in figures 2–5. $\mathrm{I}\Delta_0[\Gamma]$, $\mathrm{III}\Delta_0[\Gamma]$ and $\mathrm{IV}\Sigma_0[\Gamma]$ are the prototype approximants in the relevant series; they are a square

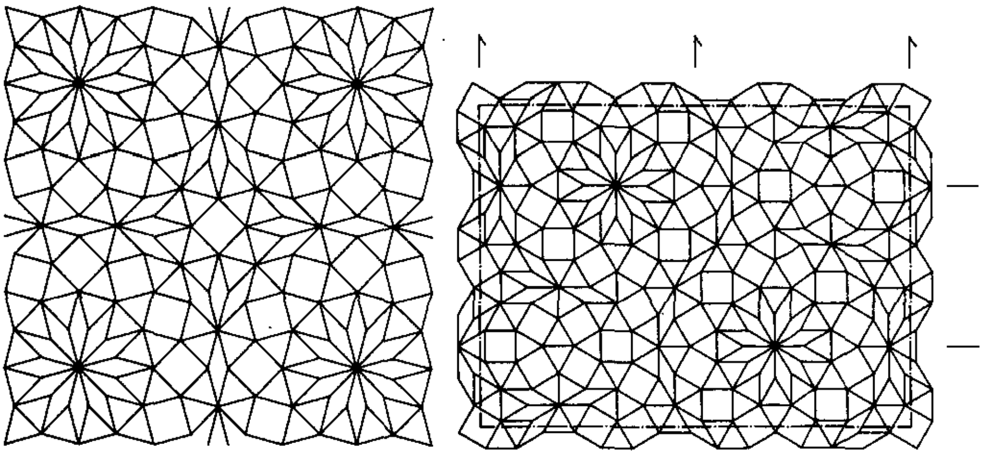
Table 2. The space groups of regular approximants in the four series I, II, III and IV. The Bravais classes of the four are shown in the first column. The second block of columns show the space groups when the phase vector $P'x$ is located on the special points of L_s (the shadow lattice) as shown in the first row; M in the row should be read as T for case III. The last column shows the ratios of the lattice constants of the rectangular unit cell.

		Γ	M	X	Y	a_2/a_1
I	p4mm	p4mm	p4g			1
II	pmm	pmm	pgg	pgm	pmg	$\sqrt{3}-1$
III	p6mm	p6mm	p31m			$\sqrt{3}$
IV	cmm	cmm	cmm			$2-\sqrt{3}$



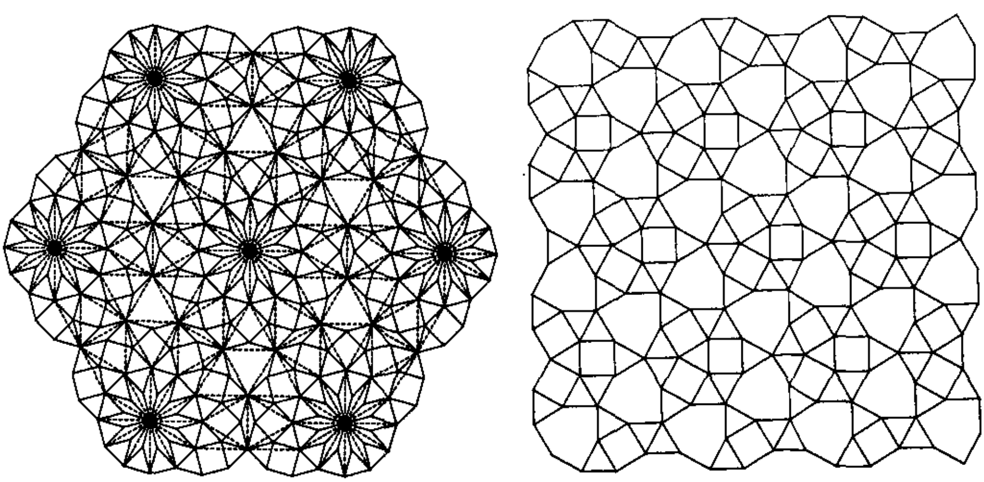
(a) (b)

Figure 2. The periodic approximants (a) $\mathrm{I}\Delta_1(\Gamma)$/p4mm and ($b$) $\mathrm{I}\Delta_1(M)$/p4g (full lines) and their inflations (broken lines). The inflated PAs are equal to $\mathrm{I}\Sigma_0(\Gamma)$ and $\mathrm{I}\Sigma_0(M)$ except for their scales. Both the two PAs with full lines are derived from the CDL $\mathrm{I}\Delta_1$. The unit cell of (a) is the square whose corners coincide with the centres of the 12-pronged vertices, while that of (b) is shown with chained lines. The vertices (or edge centres) of the unit cell in (b) are the special points of the point group 4 (or mm).



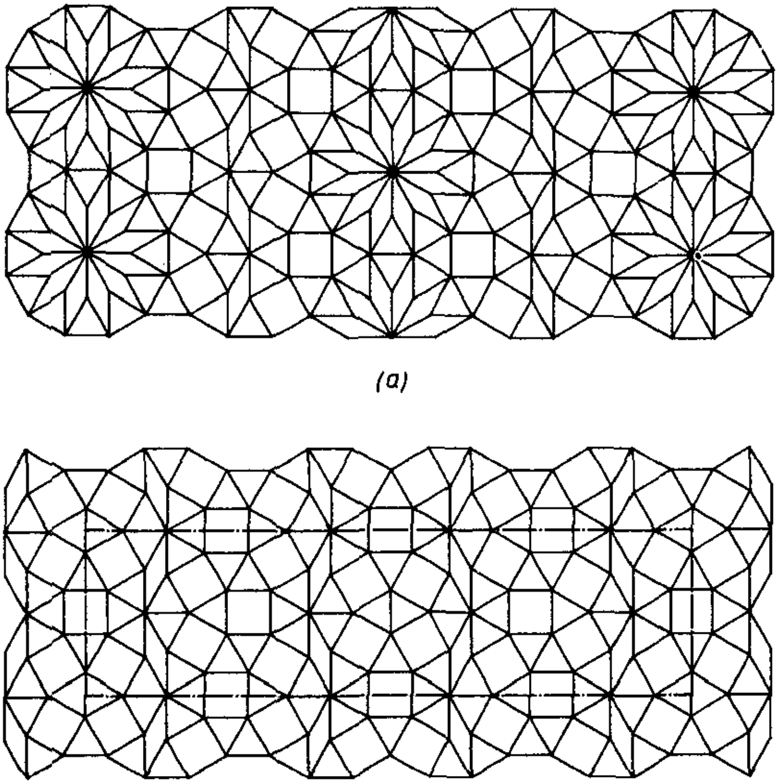
(a) (b)

Figure 3. The periodic approximants (*a*) $\mathrm{I}\Sigma_1(\Gamma)$/p4mm and (*b*) $\mathrm{II}\Delta_2(X)$/pgm. In (*a*) the PA is derived from $\mathrm{I}\Delta_1(\Gamma)$ in figure 2(*a*) by the improper DAR or from $\mathrm{I}\Sigma_0(\Gamma)$ (cf the PA in broken lines in figure 2(*a*)) by the proper one. The unit cell is the square formed by the centres of the 12-pronged vertices. In (*b*) the bars (or arrows) show the mirrors (or glides) of pgm.



(a) (b)

Figure 4. The periodic approximants (*a*) $\mathrm{III}\Sigma_1(\Gamma)$/p6mm (full lines) and (*b*) $\mathrm{III}\Delta_1(T)$/p31m. In (*a*) the centres of the 12-pronged vertices are the lattice points of the Bravais lattice. The lattice points in the six trigonal hexagons (full lines) are ignored because of the 'frustration'. The once inflated QL ($\mathrm{III}\Delta_1(\Gamma)$/p6mm) is shown by broken lines. In (*b*) the trigonal hexagons are due to the 'frustration'. The PA is considered, alternatively, to be a PA to the DQL derived with the window $W_{D'}$. The centres of the trigonal hexagons with a common orientation form the Bravais lattice of the PA.



(a)

(b)

Figure 5. The periodic approximants (*a*) $\mathrm{IV}\Delta_2(\Gamma)$/cmm and (*b*) $\mathrm{IV}\Delta_2(M)$/cmm. The Bravais lattices of (*a*) and (*b*) are identical; the rectangular unit cell is shown by the chain lines in (*b*).

lattice, a triangular one and a rhombic one, each of which is associated with the periodic tilings by only squares, triangles or rhombi in the DQL. A PA may incur symmetry breaking due to a 'frustration'; some lattice points of the mother lattice are projected on the boundaries of the window and the 'frustration' cannot be resolved without breaking the symmetry of the PA (Niizeki 1991b).

5. Discussion

We have obtained four series, I, II, III and IV, of CDLs by restricting p/q in $\langle p/q, r/s\rangle$ to $\{w_k/u_k\}$, i.e. one of the three series of best approximants to $\sqrt{3}$. We can obtain similar series from the remaining two, $\{t_k/v_k\}$ and $\{3u_k/w_k\}$. Since $\{v_k\}$ and $\{t_k\}$ are odd series, the square CDL $\langle t_k/v_k, t_k/v_k\rangle$ is reducible. In fact, it is equivalent to $\mathrm{I}\Sigma_{k-1}$ $(=\hat{\tau}_0\mathrm{I}\Delta_{k-1})$. On the contrary, $\langle 3u_k/w_k, 3u_k/w_k\rangle$ is irreducible. The lattice constant a_1 $(=a_2)$ of the relevant L_2 is $\sqrt{3}$ times that of $\mathrm{I}\Delta_k$. We shall denote the resulting series of CDLs as $\mathrm{I}'\Delta$. Similarly, series IV has two variants IV′ and IV″ and the lattice constants of L_2 of the variant CDLs are $(\sqrt{3}-1)$ and $\sqrt{3}$ times those of the corresponding CDL in IV. We can obtain, however, no variants from II or III provided that r/s is restricted to best approximants to $\sqrt{3}$. In summary, we have obtained six new series of CDLs, $\mathrm{I}'\Delta$, $\mathrm{I}'\Sigma$, $\mathrm{IV}'\Delta$, $\mathrm{IV}'\Sigma$, $\mathrm{IV}''\Delta$ and $\mathrm{IV}''\Sigma$. The space groups of the PAs in the variant series are similar to those in their originals.

The three series of PAs, $\mathrm{IV}\Delta[P]$, $\mathrm{IV}'\Delta[P]$ and $\mathrm{IV}''\Delta[P]$ with $P=\Gamma$ or M, are merged into one grand series of PAs and they are the members of the three-cycles in the grand series; the space groups are common (cmm) among the PAs in the grand series and the unit cells of the PAs are similar. It can be shown that the last two series are related to the first by the transformation associated with the type II self-similarity of the DQL. Similarly, $\mathrm{I}\Delta[\Gamma]$ and $\mathrm{I}'\Delta[\Gamma]$ are merged into a grand series with two-cycles.

The phason strain in a CDL is completely characterized by the 2×2 block S at the bottom left of Φ (see I). S is diagonal for the phason strain with two mirrors and S_{11} and S_{22} represent the magnitudes of the phason strain along the two mirror axes. A brief calculation yields that S_{11} and S_{22} of $\langle p/q, r/s\rangle$ are given by $c(\sqrt{3}\,q-p)/(p+\sqrt{3}\,q)$ and $c(r-\sqrt{3}\,s)/(r+\sqrt{3}\,s)$ with $c=a'/a$, while those of $\langle p/q, r/s\rangle_\Sigma$ are τ^{-1} times those of $\langle p/q, r/s\rangle$. S_{11} and S_{22} decrease by factor $1/\tau^{2k}$ in each series of CDLs. Naturally, $|S_{11}|=|S_{22}|$ for a square CDL or a hexagonal one; signs are different between S_{11} and S_{22} for a square CDL because $(r')^3$ $(\in H')$ is not equal to r^3 $(\in H)$ but to $-r^3$. It is interesting that the sign of S_{11} (or S_{22}) is constant in each series of CDLs, in contrast to the case of other QLs in 2D and 3D where it alternates (Niizeki 1991b, c); this is because each series of approximants to $\sqrt{3}$ tends to $\sqrt{3}$ from one side only.

We have discussed in this paper PAs to so-called Bravais-type DQLs. A non-Bravais-type QL is obtained from a Bravais-type one by an appropriate decoration (see, for example, Niizeki 1989e). The atomic structure of a real quasicrystal is based usually on a non-Bravais-type QL (Janssen 1988). There exist many non-Bravais-type DQLs (Niizeki 1988, 1989b, d, Socolar 1989, Nissen 1990, Stampfli 1990). The decagonal QL associated with Penrose's rhombic tiling is also of non-Bravais-type (Niizeki 1989b). The previous theory (Niizeki 1991a) of the space groups of PAs to a QL and the theory of their 'self-similarity' developed in I can be extended to include the non-Bravais-type QLs. This subject will be fully developed in a separate paper.

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Appendix

Three unimodular matrices (a) R, (b) M and (c) M_0:

$$\overset{(a)}{\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \qquad \overset{(b)}{\begin{pmatrix} 2 & 1 & 0 & -1 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ -1 & 0 & 1 & 2 \end{pmatrix}} \qquad \overset{(c)}{\begin{pmatrix} 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}}.$$

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